Relativistic Doppler effect

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"I think nature's imagination is so much greater than man's, she's never going to let us relax."

— Richard Feynman

In continuation of the earlier articles on special theory of relativity, we study here modified result for aberration of light in relativity and as well as relativistic Doppler effect. We compare these results with the classical cases and see that under special conditions the relativistic aberration and Doppler effect formula reduces to their respective classical forms. Meaning of most of the symbols used here are the same as was used in earlier notes.

In Fig. (1), we have considered a train of plane monochromatic wave. It originates from O' of the S' frame and waves are considered to be parallel to $x' - y'$ plane. We know that the propagating wave equation can be written as $\phi(\mathbf{r}) \sim \exp i(\mathbf{k}.\mathbf{r} - \omega t)$, where $|\mathbf{k}| = k$ is wave number related to wavelength λ as $k = 2\pi/\lambda$. This can be written in terms of Sine/Cosine forms. We consider here plane monochromatic wave of unit amplitude and for simplicity write its equation (in terms of cosine form) in the S'-frame as

$$\phi' \sim \cos 2\pi \left[\frac{x' \cos\theta' + y' \sin\theta'}{\lambda'} - \nu' t' \right], \tag{1}$$

where $\omega = 2\pi\nu$ is the frequency. Similarly, in S-frame we write

$$\cos 2\pi \left[\frac{x \cos\theta + y \sin\theta}{\lambda} - \nu t \right]. \tag{2}$$

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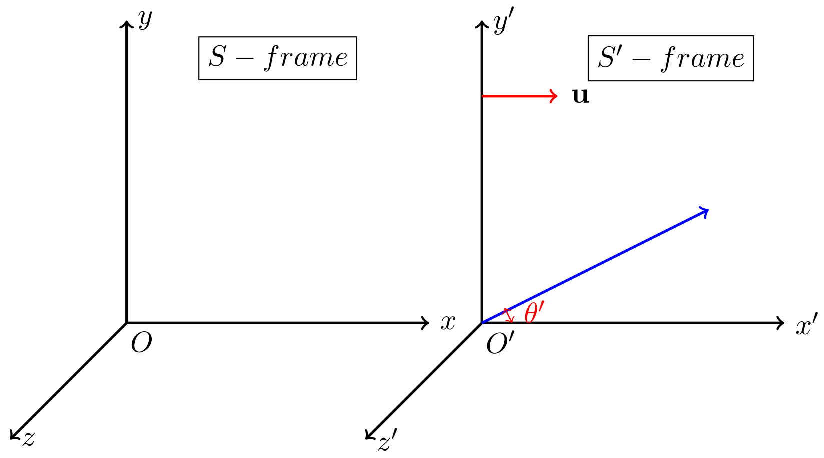

FIG. 1: S' frame moving with constant velocity $\mathbf{u}$ relative to S frame in the +ive $x-$direction, where $\mathbf{u} = (u, 0, 0)$. Both the frames have common x-x'-axis and other axes y-y' and z-z' of the two frames are parallel. Initially at $t = t' = 0$, both the frames were coinciding at the origin O. A plane monochromatic wave emitted (rays parallel to $x' - y'$ plane) from O' of the S'-frame and propagation direction makes an angle θ' with the x' axis.

It is to be noted here that velocity of light is same for both the observers so we can write

$$\lambda\nu = \lambda'\nu' = c. \tag{3}$$

To refresh memory we write here Lorentz transformation equations

$$x' = \gamma(x - ut), \quad y' = y, \quad z' = z \quad \text{and} \quad t' = \gamma(t - ux/c^2), \tag{4}$$

where $\gamma = 1/\sqrt{1-\beta^2}$ with $\beta = u/c$. Now using the above transformation we can write left-hand side (lhs) of Eq. (1) as

$$\cos 2\pi \left[\frac{\gamma(x - ut)\cos\theta' + y \sin\theta'}{\lambda'} - \gamma\nu'(t - ux/c^2) \right]. \tag{5}$$

Further, the above equation can be be rewritten as

$$\cos 2\pi \left[\frac{\gamma(\cos\theta' + u/c)}{\lambda'} x + \frac{y\sin\theta'}{\lambda'} - \gamma\nu'(1 + (u/c)\cos\theta')t \right] \tag{6}$$

or

$$\cos 2\pi \left[\frac{(\cos\theta' + \beta)}{\lambda'\sqrt{1-\beta^2}} x + \frac{\sin\theta'}{\lambda'} y - \frac{\nu'(1 + \beta\cos\theta')}{\sqrt{1-\beta^2}} t \right] \tag{7}$$

Now comparing Eq. (7) with Eq.(2), we obtain

$$\frac{\cos\theta}{\lambda} = \frac{(\cos\theta' + \beta)}{\lambda'\sqrt{1-\beta^2}}, \tag{8}$$

$$\frac{\sin\theta}{\lambda} = \frac{\sin\theta'}{\lambda'}, \tag{9}$$

$$\nu = \frac{\nu'(1+\beta\cos\theta')}{\sqrt{1-\beta^2}}. \tag{10}$$

From Eqs. (8) and (9), we obtain the **relativistic aberration formula** *which is the correction of classical result.*

$$\tan\theta = \frac{\sin\theta'\sqrt{1-\beta^2}}{\beta + \cos\theta'}. \tag{11}$$

Note that for $u \ll c$ it reduces to the classical result.

Home work: *Derive the relativistic aberration formula using velocity transformation rules.*

The result (11) can be obtained for primed frame observer S' by simply interchanging primed quantities to unprimed and replacing $\beta(= u/c) \to -\beta$.

$$\tan\theta' = \frac{\sin\theta\sqrt{1-\beta^2}}{\cos\theta - \beta}. \tag{12}$$

Equation for Doppler effect in relativity

The relativistic formula for Doppler effect is given by Eq. (10) obtained earlier. We again write here to discuss special cases of the equation.

$$\nu = \frac{\nu'(1+\beta\cos\theta')}{\sqrt{1-\beta^2}}. \tag{13}$$

Now for S' observer, replacing $\beta \to -\beta$ and interchanging primed quantities to unprimed in the above result, we obtain

$$\nu' = \frac{\nu(1-\beta\cos\theta)}{\sqrt{1-\beta^2}}, \tag{14}$$

or we can write

$$\nu = \frac{\nu'\sqrt{1-\beta^2}}{1-\beta\cos\theta}, \tag{15}$$

Now we discuss following cases of Eq. (13):

Case-I *Longitudinal Doppler effect.* We take $\theta = 0$ or $\theta = 180^0$ for longitudinal Doppler effect. $\theta = 0$ indicates that source and observer are moving towards each other and $\theta = 180^0$

indicates that source and observer are moving away from each other. In Eq. (15) we take $\theta = 0$ and obtain,

$$\begin{aligned} \nu &= \frac{\nu' \sqrt{1-\beta^2}}{1-\beta} \\ &= \nu' \sqrt{\frac{1+\beta}{1-\beta}} \\ &= \nu' \sqrt{\frac{c+u}{c-u}}. \end{aligned} \tag{16}$$

Similarly for $\theta = 180^0$, we obtain

$$\nu = \nu' \sqrt{\frac{c-u}{c+u}}. \tag{17}$$

Case-II *Transverse Doppler effect.* Transverse Doppler effect (motion of source and observer are transverse to each other) is obtained by setting $\theta = 90^0$ in Eq. (15).

$$\nu = \nu' \sqrt{1-\beta^2}. \tag{18}$$

Thus, in relativity we see that in transverse motion there is also change in the observed frequency. Here we see that the observed frequency ν is lower than the proper frequency ν' of source.

Classical result: From Eq. (15), we can obtain the classical formula for the Doppler effect. We take $\beta \ll 1$ and use binomial expansion of the equation

$$\begin{aligned} \nu &= \frac{\nu' \sqrt{1-\beta^2}}{1-\beta\cos\theta} \\ &\simeq \nu'(1-\beta\cos\theta)^{-1} \\ &= \nu'(1+\beta\cos\theta) \end{aligned} \tag{19}$$

Now we discuss the two cases of the above equation.

Case-I *Longitudinal.* As discussed earlier, longitudinal effect is obtained by setting $\theta = 0$ or $\theta = 180^0$. For $\theta = 0$ in Eq. (19),

$$\nu = \nu'(1 + u/c),$$

which indicates that observed frequency is greater than proper frequency. For $\theta = 180^0$,

$$\nu = \nu'(1 - u/c),$$

indicating observed frequency is lower than the proper frequency.

Case-II *Transverse.* For transverse effect we set $\theta = 90^0$ in Eq. (19), and obtain $\nu = \nu'$. Thus, we see that there is no Doppler effect in transverse motion of the source and observer in the classical case.

[1] R. Resnick, *Introduction to Special Relativity*, Wiley-VCH, (1968) .

[2] A. Beiser, *Concepts of modern physics*, Tata McGraw-Hill Education, (2003).